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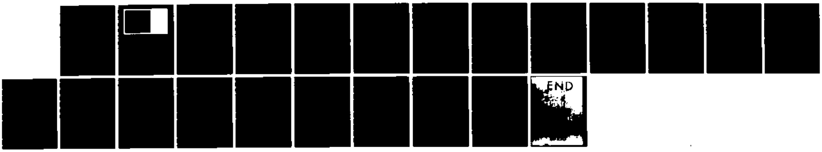

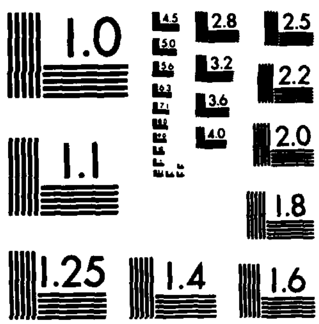

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RISING BUBBLES IN A TWO-DIMENSIONAL TUBE WITH SURFACE TENSION

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ABSTRACT

The motion of a two-dimensional bubble rising at a constant velocity U in a tube of width h is considered. The acceleration of gravity g and the surface tension T are included in the dynamic boundary condition. For $T = 0$, the results of Garabedian and Vanden-Broeck show that a solution exists for all values of the Froude number $F = U/(gh)^{1/2}$ (raised to the ½ power) smaller than a critical value $F_c \approx 0.36$. In this paper accurate numerical solutions with $T \neq 0$ (unequal to) are computed by series truncations. The results show that for each value of $T \neq 0$, there exists a countably infinite number of solutions. Each of these solutions corresponds to a different value of F. As T tends to zero, all these solutions approach a unique limiting solution characterized by $F = F^* \approx 0.23$. Therefore, the degenerancy associated with $T = 0$ is removed by including the effect of surface tension. In addition the profile corresponding to $F = F^*$ is found to be in good agreement with experimental data.

AMS(MOS) Subject Classification: 76B10

Key Words: bubble, surface tension

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SIGNIFICANCE AND EXPLANATION

The motion of a two-dimensional bubble rising at a constant velocity U in a tube of width h is considered (see Figure 1). The acceleration of the gravity g and the surface tension T are included in the formulation of the problem. For $T = 0$, Garabedian[1] and Vanden-Broeck[2] have shown that a solution exists for all values of F smaller than a critical value F_c. Garabedian[1] used a stability argument to suggest that the only physically acceptable solution is the one corresponding to $F = F_c$. Vanden-Broeck[2] solved the problem with $T = 0$ numerically. He found that $F_c = 0.36$. This value is about forty percent higher than the experimental value of 0.25 of Collins[3].

In the present paper we show that the discrepancy between the theoretical value 0.36 and the experimental value 0.25 is considerably reduced by taking into account the effect of surface tension. We show that for each value of $T \neq 0$, there exists a countably infinite number of solutions. As T tends to zero, all these solutions approach a unique limiting configuration characterized by $F = F^* \sim 0.23$.

These results imply that the physically revelant solution, when $T = 0$, is not the solution corresponding to $F = F_c \sim 0.36$ but the soltuion corresponding to $F = F^* \sim 0.23$.

RISING BUBBLES IN A TWO-DIMENSIONAL TUBE WITH SURFACE TENSION

Jean-Marc Vanden-Broeck

I. Introduction

We consider a bubble rising at a constant velocity U in a two-dimensional tube of width h. We choose a frame of reference moving with the bubble and we assume that the bubble extends downwards without limit (see Fig. 1.)

As we shall see the problem is characterized by the Froude number

$$F = \frac{U}{(gh)^{1/2}} \tag{1}$$

and the Weber number

$$\alpha = \frac{\rho U^2 h}{T} \tag{2}$$

Here g is the acceleration of gravity, T the surface tension and ρ the density of the fluid.

Garabedian [1] considered the problem with $T = 0$ (i.e. $\alpha = \infty$). He presented analytical evidence that a solution exists for all values of F smaller than a critical value F_c. He then used an energy argument to suggest that the only physically significant solution is the one for which $F = F_c$. In addition he showed that $F_c > 0.2363$ and guessed the value $F_c = 0.24$. Birkhoff and Carter [2] and Collins [3] using different approaches obtained the value 0.23.

Vanden-Broeck [4] solved the problem with $T = 0$ numerically. His results confirm that a solution exists for all values of F smaller than a

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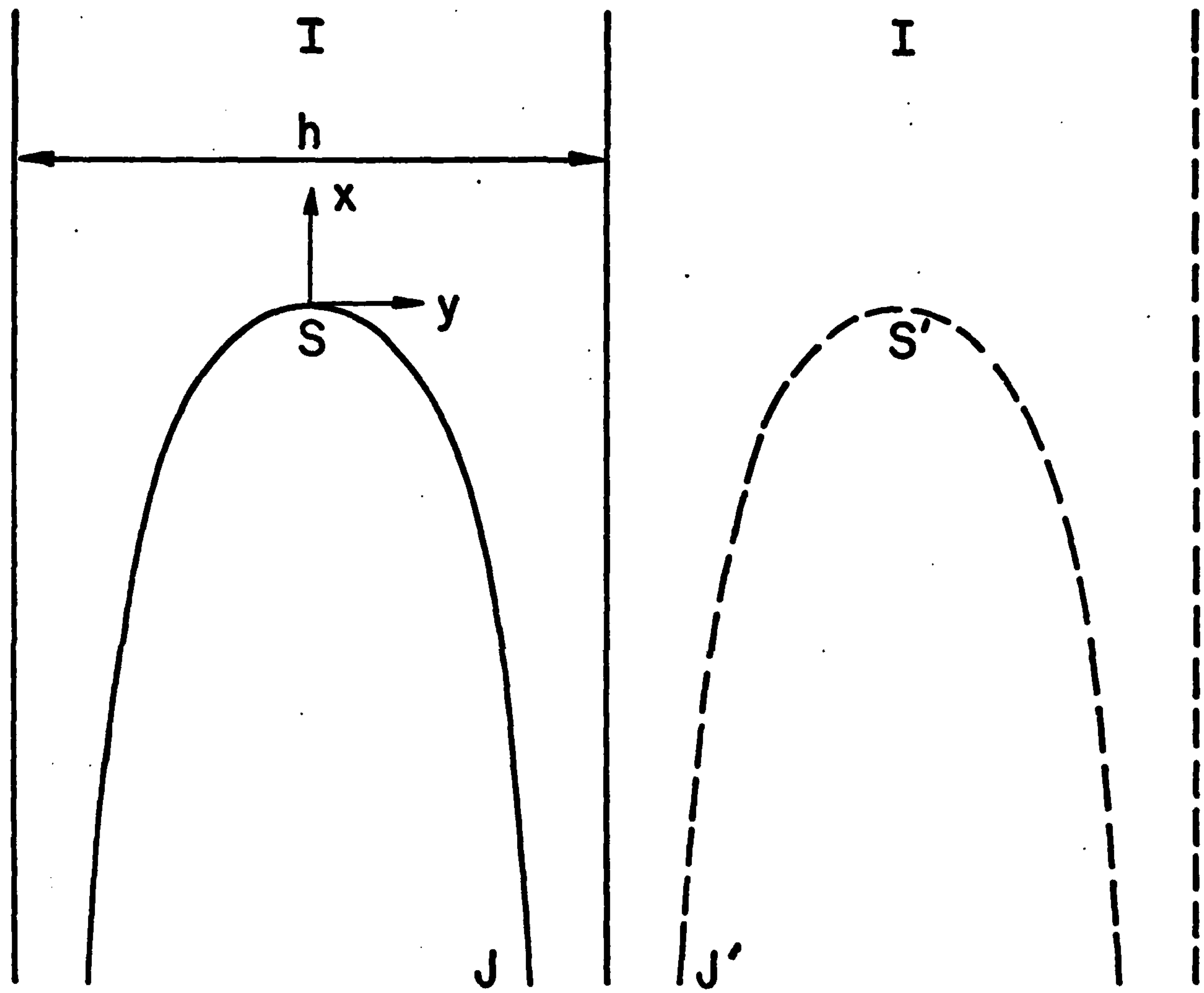

Figure 1

Sketch of the flow and the coordinates. The broken lines correspond to the reflexion of the flow domain into the wall $y = \frac{1}{2}$.

critical value F_c. However Vanden-Broeck's results do not support the approximate calculations of Garabedian [1], Birkhoff and Carter [2] and Collines [3]. Vanden-Broeck found $F_c = 0.36$. This value is about forty percent higher than the experimental value 0.25 presented by Collins [3].

In the present paper we show that the discrepancy between the experiemntal value $F = 0.25$ and the theoretical value $F_c = 0.36$ is considerably reduced by taking into account the effect of surface tension. We show that for each value of $T \neq 0$, there exists a countably infinite number of solutions. Each of these solutions corresponds to a different value of F. As T tends to zero, all these solutions approach a unique limiting configuration characterized by $F = F^* \sim 0.23$. The corresponding profile is found to be in good agreement with the experimental data of Collins [3].

It is interesting to note that the present inviscid problem is qualitatively similar to the problem of viscous flow in a Hele-Shaw cell (Saffman and Taylor [5], McLean and Saffman [6], Vanden-Broeck [7]). Both problems are characterized by a continuum of solutions for $T = 0$ and a discrete set of solutions for $T > 0$.

The problem is formulated in Sec. II. The numerical procedure is described in Sec. III and the results are discussed in Sec. IV.

II. Formulation

Let us consider the steady two-dimensional potential flow of an inviscid incompressible fluid past a bubble in a tube of width h (see Fig. 1). The pressure in the bubble is assumed to be constant. We introduce cartesian coordinates with the origin at the top of the bubble and we assume that the bubble is symmetric about the x-axis. Gravity acts in the negative x-direction. As $x \to \infty$, the velocity approaches the constant U.

We define dimensionless variables by taking U as the unit velocity and h as the unit length. We denote the potential function by ϕ and the stream function by ψ. In addition we introduce the complex velocity by $\zeta = u - iv$ and we define the function $\tau - i\theta$ by the relation

$$\zeta = u - iv = e^{\tau - i\theta} \tag{3}$$

Without loss of generality we choose $\phi = 0$ at $x = y = 0$ and $\psi = 0$ on the surface of the bubble. It follows from the choice of the dimensionless variables that $\psi = -\frac{1}{2}$ on the wall $y = \frac{1}{2}$.

We satisfy the kinemetic condition on the wall $y = \frac{1}{2}$, by reflecting the flow in that boundary (see Fig. 1). We shall seek the complex function $\tau - i\theta$ as an analytic function of $f = \phi + i\psi$ in the strip $-1 < \psi < 0$. The complex plane is sketched in Fig. 2.

On the surface of the bubble, the Bernoulli equation yields

$$1/2\, q^2 + gx - \frac{T}{\rho}K = B \quad \text{on } SJ \text{ and } S'J'. \tag{4}$$

Here q is the flow speed, K the curvature of the free surface and B the Bernoulli constant. In dimensionless variables (4) becomes

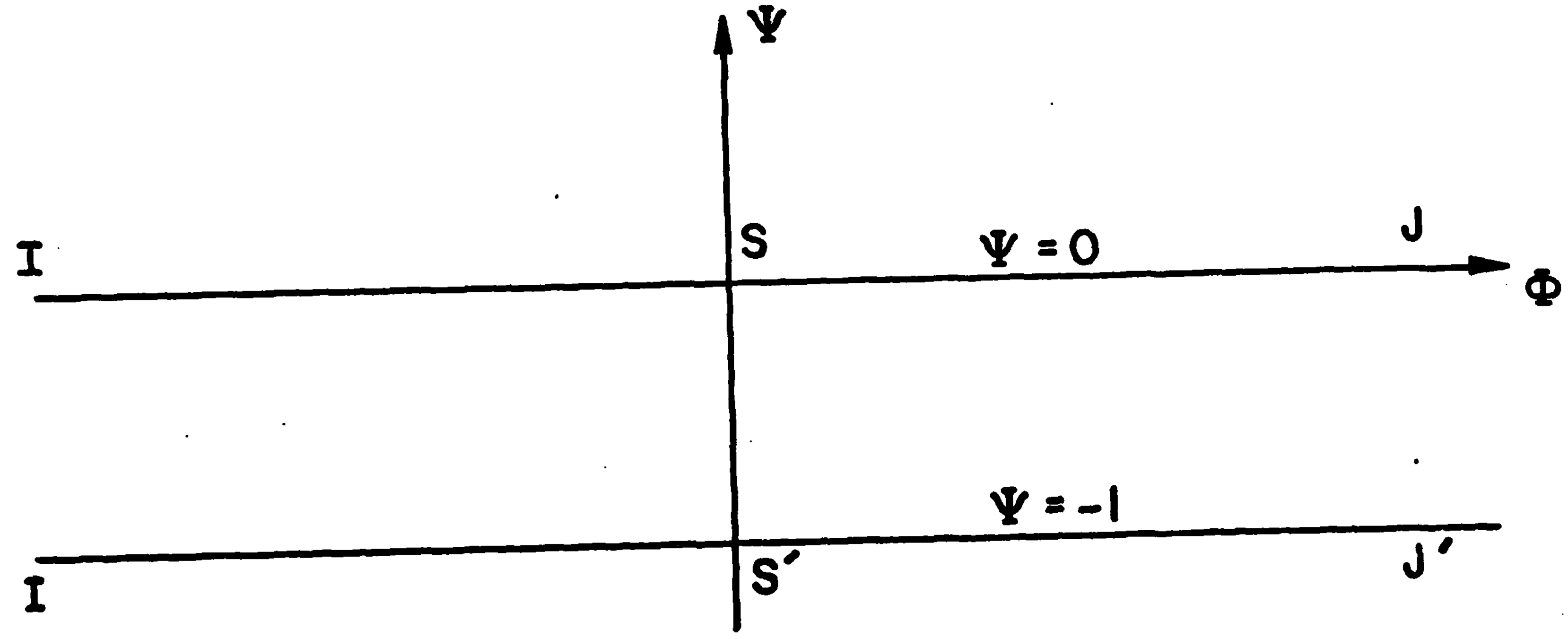

Figure 2

Flow configuration in the complex potential plane.

$$e^{2\tau} + \frac{2}{F^2}x - \frac{2}{\alpha}e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| = \frac{2B}{U^2}, \quad \text{on SJ and S'J'}. \qquad (5)$$

Here F and α are the Froude number and the Weber number defined by (1) and (2) respectively.

It is convenient to eliminate x and B from (5) by differentiating (5) with respect to ϕ. Using the relation

$$\frac{\partial x}{\partial\phi} + i\frac{\partial y}{\partial\phi} = \frac{1}{u-iv} = e^{-\tau+i\theta} \qquad (6)$$

we obtain

$$e^{2\tau}\frac{\partial\tau}{\partial\phi} + \frac{1}{F^2}e^{-\tau}\cos\theta - \frac{1}{\alpha}\frac{\partial}{\partial\phi}\left(e^{\tau}\frac{\partial\theta}{\partial\phi}\right) = 0 \quad \text{on SJ and S'J'}. \quad (7)$$

The kinemetic conditions on IS and IS' yield

$$\theta = 0, \quad \psi = -1 \quad \phi < 0 \qquad (8)$$

$$\theta = 0, \quad \psi = 0 \quad \phi < 0. \qquad (9)$$

Finally the flow configuration of Fig. 1 is characterized by a stagnation point and a horizontal slope at the top of the bubble. This yields the additional conditions

$$\tau = -\infty, \ \theta = \frac{\pi}{2} \quad \text{at} \quad \phi = 0, \ \psi = 0 \qquad (10)$$

$$\tau = -\infty, \ \theta = -\frac{\pi}{2} \quad \text{at} \quad \phi = 0, \ \psi = -1. \qquad (11)$$

This completes the formulation of the problem of determining $\tau - i\theta$. This function must be analytic in the strip $-1 < \psi < 0$ and satisfy the conditions (7)-(11).

III. Numerical Procedure

We solve the problem by using the procedure derived by Vanden-Broeck [7] to investigate the effect of surface tension on the shape of fingers in a Hele-Shaw cell. Thus we define a modified problem by replacing (10) and (11) by

$$\tau = -\infty \quad \theta = \gamma \quad \text{at} \quad \phi = 0, \ \psi = 0 \tag{12}$$

$$\tau = -\infty \quad \theta = -\gamma \quad \text{at} \quad \phi = 0, \ \psi = -1. \tag{13}$$

Here γ is a parameter to be found as part of the solution.

We will solve the modified problem defined by (7)-(9), (12) and (13), and obtain solutions for all values of F and α. Then we will obtain the solutions of the original problem by selecting among the solutions of the modified problem those for which $\gamma = \frac{\pi}{2}$.

Following Birkhoff and Carter [2] and Vanden-Broeck [7] we define the new variable t by the relation

$$e^{-\pi f} = \frac{1}{2}\left(t + \frac{1}{t}\right) \tag{14}$$

This transformation maps the flow domain onto the unit circle in the complex t-plane (see Fig. 3).

We note that

$$\zeta \sim [\ell n(1 \pm it)]^{1/3} \quad \text{as} \quad t \to \pm i$$

$$\zeta \sim (1 \pm t)^{\gamma\, 2/\pi} \quad \text{as} \quad t \to \mp 1$$

(see Birkhoff and Carter [2] for details). Therefore we define the function $\Omega(t)$ by the relation

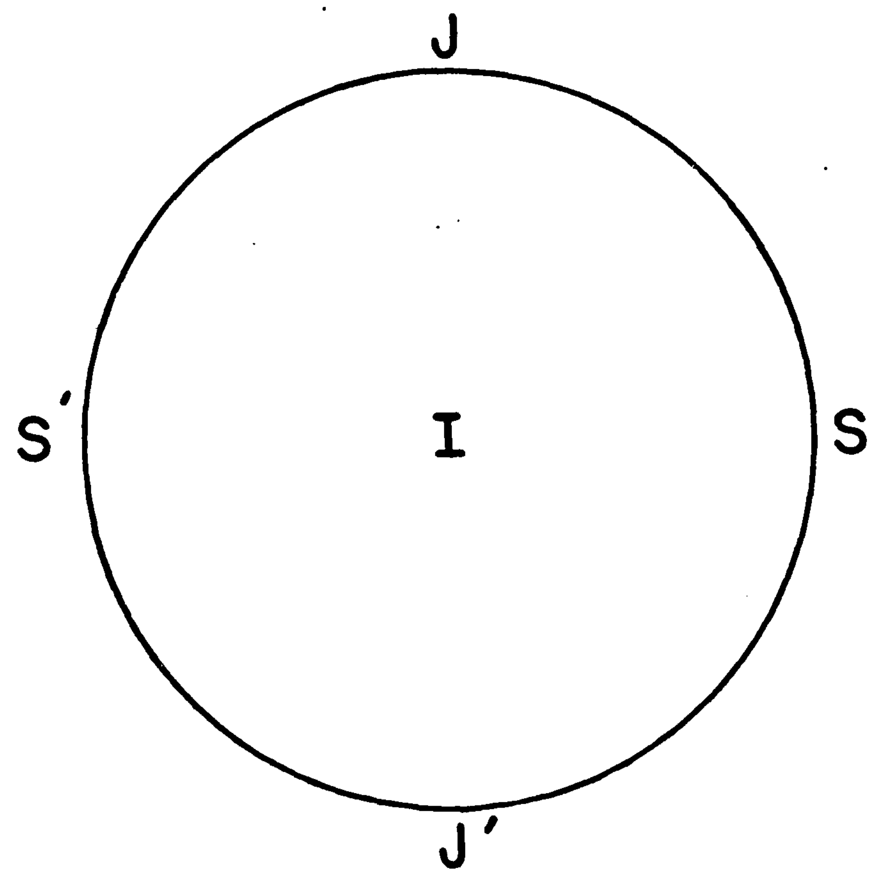

Figure 5

The complex t-plane.

$$\zeta = -[-\ell n\, C(1+t^2)]^{1/3}(-\ell n\, C)^{-1/3}(1-t^2)^{\gamma\, 2/\pi}[1+\Omega(t)]. \qquad (15)$$

Here C is an arbitrary constant between 0 and 0.5. We choose $C = 0.2$. The function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior. The conditions (8) and (9) show that $\Omega(t)$ can be expanded in the form of a Taylor expansion in even powers of t. Hence

$$e^{\tau-i\theta} = -[\ell n\, C(1+t^2)^{1/3}(-\ell n\, C)^{-1/3}(1-t^2)^{\gamma\, 2/\pi}(1+\sum_{n=1}^{\infty} a_n t^{2n}) \qquad (16)$$

The function (16) satisfy (8), (9), (12) and (13). The coefficients a_n and the parameter γ have to be determined to satisfy (7) on SJ. The condition (7) on S'J' will then be automatically satisfied by symmetry.

We use the notation $t = |t|e^{i\sigma}$ so that points on SJ are given by $t = e^{i\sigma}$, $0 < \sigma < \frac{\pi}{2}$. Using (14) we rewrite (7) in the form

$$\pi \cot g\sigma e^{2\tilde{\tau}} \frac{d\tilde{\tau}}{d\sigma} + \frac{1}{F^2} e^{-\tilde{\tau}} \cos\tilde{\theta} - \frac{1}{\alpha}\pi^2 \cot g\sigma \frac{d}{d\sigma}\left(e^{\tilde{\tau}} \cot g\sigma \frac{d\tilde{\theta}}{d\sigma}\right) = 0 \qquad (17)$$

Here $\tilde{\tau}(\sigma)$ and $\tilde{\theta}(\sigma)$ denote the values of τ and θ on the free surface SJ.

We solve the problem approximately by truncating the infinite series in (16) after $N-1$ terms. We find the $N-1$ coefficients a_n and the parameter γ by collocation. Thus we introduce the N mesh points

$$\sigma_I = \frac{\pi}{2N}\left(I-\frac{1}{2}\right), \qquad I = 1,\ldots,N. \qquad (18)$$

Using (16) and (18) we obtain $[\tilde{\tau}(\sigma)]_{\sigma=\sigma_I}$, $[\tilde{\theta}(\sigma)]_{\sigma=\sigma_I}$, $[\frac{d\tilde{\tau}}{d\sigma}]_{\sigma=\sigma_I}$, $[\frac{d\tilde{\theta}}{d\sigma}]_{\sigma=\sigma_I}$ and $[\frac{d^2\tilde{\theta}}{d\sigma^2}]_{\sigma=\sigma_I}$ in terms of the coefficients a_n and the parameter γ.

Substituting these expressions into (16) we obtain N nonlinear equations for the N unknowns a_n, $n = 1,\ldots,N-1$ and γ. We solve this system by Newton's method. Once this system is solved for a given value of F, the shape of the jet is obtained by numerically integrating (6).

IV. Discussion of the results

We use the numerical scheme described in Sec. III to compute solutions of the modified problem for various values of F and α. The coefficients a_n were found to decrease rapidly a n increases. For example $a_{10} \sim 10^{-2}$, $a_{20} \sim 2.10^{-3}$, $a_{30} \sim 3.10^{-4}$ for $F^2 = 0.02$ and $\alpha = 10$. Most of the calculations were done with $N = 50$.

In Figure 4 we present values of γ versus F for $\alpha = 10$. As F tends to infinity, γ tends to zero. As F approaches zero, γ oscillates infinitely often around $\frac{\pi}{2}$. Figure 4 shows that there exists a countably infinite number of values of F for which $\gamma = \frac{\pi}{2}$. The solutions corresponding to these values of F are the solutions of the original problem. Similar results were found for other values of α.

Vanden-Broeck solved the problem with $\alpha = \infty$, (i.e., $T = 0$). He found

$$\gamma = \frac{\pi}{2}, \quad F < F_c \sim 0.36 \tag{19}$$

$$\gamma = 0, \quad F > F_c \sim 0.36. \tag{20}$$

Therefore all the solutions corresponding to $F < F_c$ are solutions of the modified problem when $\alpha = \infty$. The solution defined by (19) and (20) is shown in Fig. 4.

The flow configuration of Fig. 1 can also serve to model a jet emerging from a nozzle whose walls consist of the streamlines IS and IS' (see Vanden-Broeck [2]). Relation (20) shows that the flow leaves the wall tangentially for $\alpha = \infty$ and $F > F_c$. On the other hand Fig. 4 indicates that the flow does not leave the walls tangentially for $\alpha = 10$ unless $F = \infty$. Similar discontinuities in slope were encountered before by Vanden-Broeck [6,7] in his analysis of the effect of surface tension on free-streamline

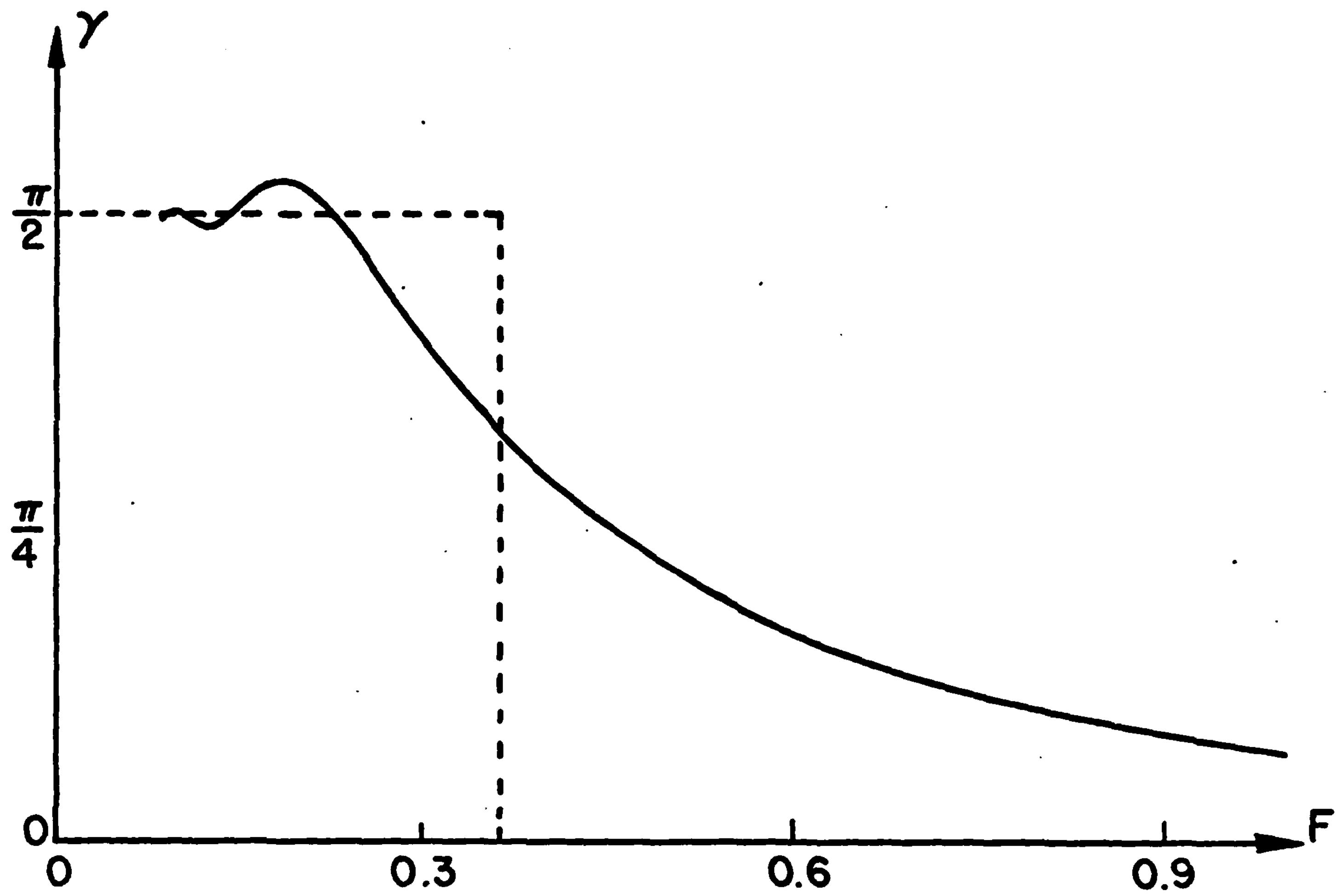

Figure 4

Values of γ versus F for $\alpha = 10$. The broken line corresponds to the solution defined by (19) and (20).

flows. The recent work of Vanden-Broeck [10] shows that these unphysical discontinuities could be removed by taking into account the thickness of the walls and the finite curvature at the ends of the walls.

As α increases, the amplitudes and the wavelengths of the oscillations in Figure 4 decrease. In Fig. 5 we present the two largest values of F, which correspond to solutions of the original problem, as functions of α^{-1}. The two curves approach $F = F^* \sim 0.23$ as α tends to infinity.

Our numerical results suggest the following general result. If n denotes an arbitrary positive integer, then the n^{th} largest value of F, which corresponds to a solution of the original problem, approaches $F = F^* \sim 0.23$ as $\alpha \to \infty$. Therefore the degenerancy associated with $T = 0$ is removed by solving the problem with $T \neq 0$ and then by taking the limit as $T \to 0$.

These results imply that the physically relevent solution, when $T = 0$, is not the solution corresponding to $F = F_c \sim 0.36$ but the solution corresponding to $F = F^* \sim 0.23$. The profile corresponding to $F = F^*$ and $T = 0$ is shown in Figure 6.

Collins[3] found the experimental value $F = 0.25$. In addition he measured the ratio of the radius of curvature at the top of the bubble to the width h of the tube and obtained the value 0.305. The corresponding ratio for the theoretical profile of Fig. 6 is 0.32.

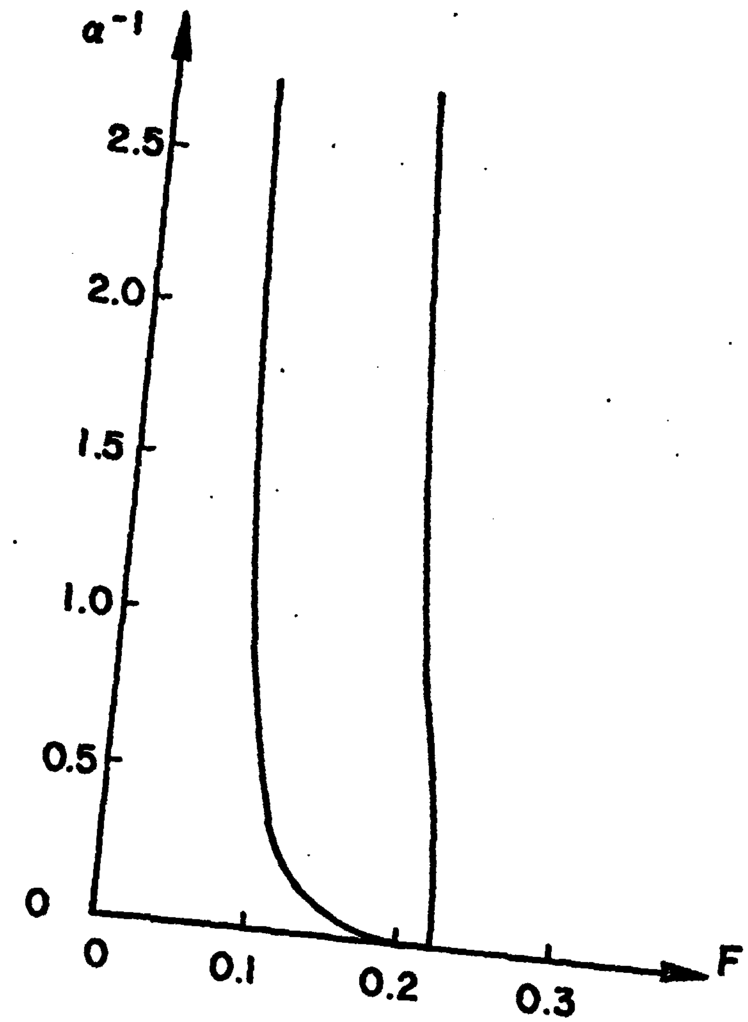

Figure 5

The two largest values of F, for which the flow configuration of Figure 1 exists, as functions of α^{-1}.

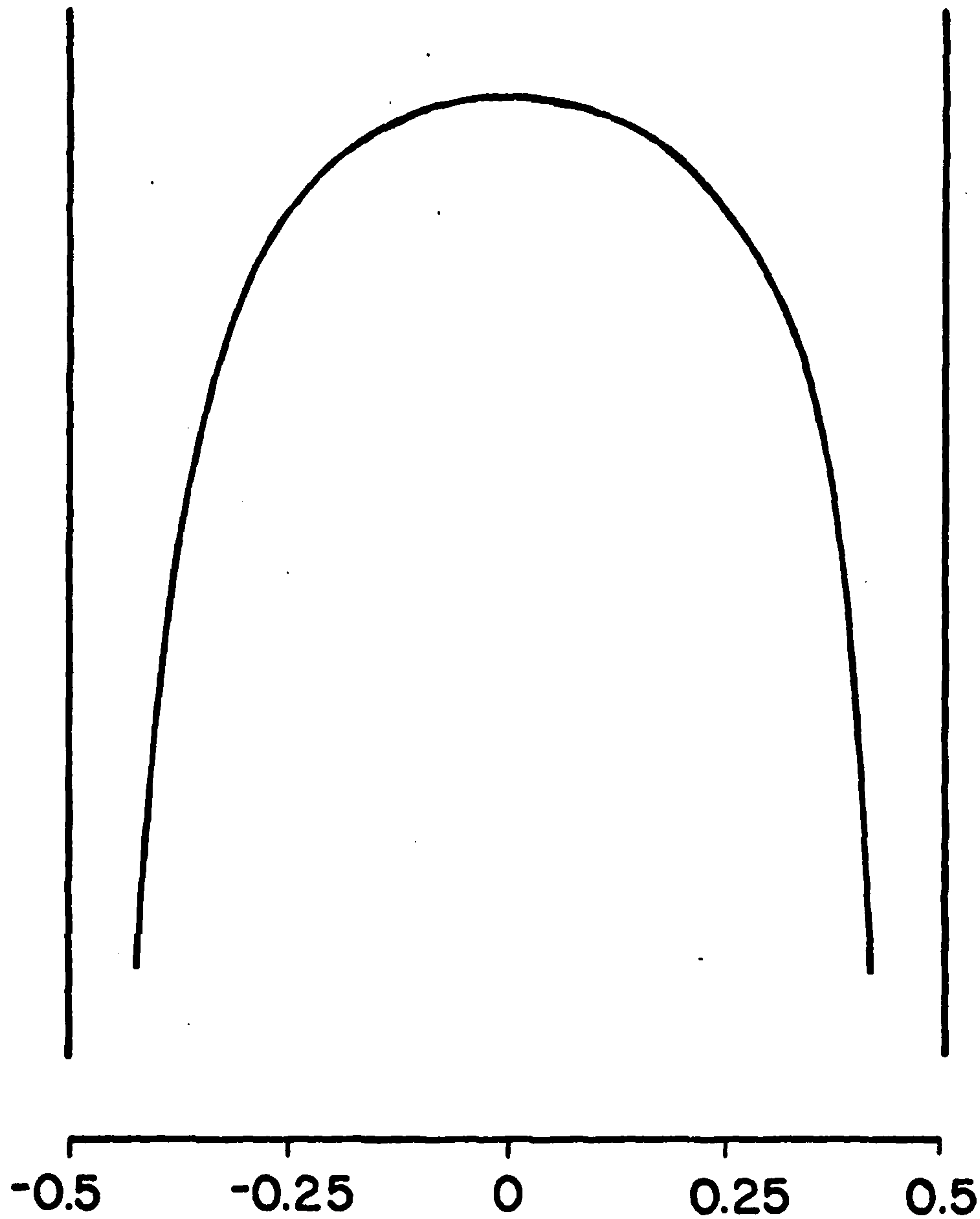

Figure 6

Bubble profile for $F = F^* \sim 0.23.$ and $T = 0.$

References

[1] P. R. Garabedian, Proc. R. Soc. London, Ser A241, 423 (1957).

[2] G. Birkhoff and D. Carter, J. of Mathematics and Physics, 6, 769 (1957).

[3] R. Collins, J. Fluid Mech., 22, 763 (1965).

[4] J.-M. Vanden-Broeck, Phys. Fluids 27, 1090 (1984).

[5] P. G. Saffman and G. I. Taylor, Proc. R. Soc. London Ser. A 245, 312 (1958).

[6] J. W. McLean and P. G. Saffman, J. Fluid Mech. 102, 455 (1981).

[7] J.-M. Vanden-Broeck, Phys. Fluids, 26, 2033 (1983).

[8] J.-M. Vanden-Broeck, J. Fluid Mech., 133, 255 (1983).

[9] J.-M. Vanden-Broeck, Phys. Fluids (in press).

[10] J.-M. Vanden-Broeck, Phys. Fluids (in press).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (cont.)

exists for all values of the Froude number $F = U/(gh)^{1/2}$ smaller than a critical value $F_c \sim 0.36$. In this paper accurate numerical solutions with $T \neq 0$ are computed by series truncations. The results show that for each value of $T \neq 0$, there exists a countably infinite number of solutions. Each of these solutions corresponds to a different value of F. As T tends to zero, all these solutions approach a unique limiting solution characterized by $F = F^* \sim 0.23$. Therefore, the degenerancy associated with $T = 0$ is removed by including the effect of surface tension. In addition the profile corresponding to $F = F^*$ is found to be in good agreement with experimental data.

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